SPHERE RECOGNITION LIES IN NP

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Abstract. We prove that the three-sphere recognition problem lies in the complexity class NP. Our work relies on Thompson's original proof that the problem is decidable [Math. Res. Let., 1994], Casson's version of her algorithm, and recent results of Agol, Hass, and Thurston [ArXiv, 2002].

1. Introduction

The three-sphere recognition problem asks: given a triangulation T, is the underlying space $|T|$ homeomorphic to the three-sphere? Clearly there are trivial points to check, such as whether $|T|$ is a three-manifold and if $|T|$ is closed, connected, orientable, *etc.* After dealing with these issues:

Theorem 15.1(Thompson [18], Casson [3]). *There is an exponential time algorithm which, given a triangulation T, decides whether or not $|T|$ is homeomorphic to the three-sphere.*

Our goal is to show:

Theorem 16.1. *The three-sphere recognition problem lies in the complexity class* **NP**.

That is, if T is a triangulation of the three-sphere then there is a polynomial sized proof of this fact. The essential details of such a proof is called a *certificate.*

As a corollary of Theorem 16.1:

Corollary 1.1. *The three-ball recognition problem lies in the complexity class* **NP**.

To prove Corollary 1.1 we must produce a certificate for every triangulation T of the three-ball. (See Section 4 for how to verify, in polynomial time, that T is a three-manifold.) First write down a polynomial sized proof that $S = \partial T$ is two-sphere. Next we give the certificate that $D(T)$, the *double* of T, is a three-sphere using Theorem 16.1. This

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completes the certificate as, by Alexander's Theorem (see Theorem 1.1 of [10]), the two-sphere S cuts $D(T)$ into a pair of three-balls.

Theorem 16.1 could be used to place other problems in **NP**: for example, the problem of deciding whether or not a triangulated three-manifold is the complement of a knot in S^3. Jaco and Sedgwick [14] show that in any nontrivial knot complement there is a meridional boundary slope which is not too long. Combining this with our work and a discussion on layered solid tori gives a proof.

It is possible that these techniques also show that the lens space recognition and surface bundle recognition problems lie in **NP**. (See Section 17.) Corollary 1.1 may also find application for certifying that a triangulated three-manifold is Haken.

Before abandoning the introduction we mention that many other problems in three-manifold topology lie in **NP**. Hass, Lagarias, and Pippenger [9] have shown that the unknotting problem, first solved by Haken, lies in **NP**. Agol, Hass and Thurston [1] have shown that the 3-manifold knot genus problem is in fact **NP**-complete. (The 3-manifold knot genus problem asks; given a genus g and a knot K in the one-skeleton of a triangulated three-manifold M, is there an orientable spanning surface for K with genus at most g?) Also, S. Ivanov [12] has shown that the three-sphere recognition algorithm is in **NP** when the triangulations considered are assumed to be *zero-efficient*. (See Remark 2.1.)

In this paper I rely heavily on material discussed in [9] and [1] as well as work of Jaco and Rubinstein [13] and a talk of Casson's at MSRI [3]. I thank Ian Agol for suggesting that I tackle this problem and for suggesting that ideas from my thesis [17] might be useful in its solution.

2. Sketch of the main theorem

Before diving into the details of the definitions we give a sketch of Theorem 16.1. We closely follow Casson's algorithm for recognizing the three-sphere.

Fix T, a triangulation of S^3. Produce a certificate $\{(T_i, v(S_i))\}_{i=0}^n$ as follows: The triangulation T_0 is equal to T. For every i use Lemma 5.9 to find S_i, a normal two-sphere in T_i which is not vertex linking, if such exists. If T is zero-efficient then Lemma 5.9 provides S_i, an almost normal two-sphere in T_i. (Here $v(S_i)$ is a *surface vector*; a concise representation of S_i. See Section 5 for the definition.)

If S_i is normal apply Theorem 14.1: T_{i+1} is obtained from T_i by crushing T_i along S_i. Briefly, we cut $|T_i|$ along S_i, cone the resulting two-sphere boundary components to points, and collapse non-tetrahedral cells of the resulting cell structure to obtain the triangulation T_{i+1}. This is discussed in Section 14, below.

If S_i is almost normal then obtain T_{i+1} from T_i by deleting the component of $|T_i|$ which contains S_i. Finally, the last triangulation T_n is empty, as is S_n.

That completes the construction of the certificate. We now turn to the procedure for checking a given certificate: that is, we cite a series of polynomial time algorithms which verify each part of the certificate. First, check if $T = T_0$ using Theorem 4.4. Next, verify that S_i is in fact the desired surface by checking its Euler characteristic (see Lemma 5.2) and checking that it is connected (see Theorem 5.3). Next, if S_i is normal verify that the triangulation T_{i+1} is identical to the triangulation obtained by crushing T_i along S_i. This uses Theorem 14.1 and again uses Theorem 4.4. If S_i is almost normal then check that the component T' of T_i containing S_i has $|T'| \cong S^3$ using Theorem 13.1 and Theorem 11.3.

Finally, by Theorem 14.2, for every i we have that $\#|T_i| \cong \#|T_{i+1}|$ where the connect sum on the left hand side ranges over the components of $|T_i|$ while the right hand side ranges over the components of $|T_{i+1}|$. By definition the empty connect sum is S^3, and this finishes the verification of the certificate.

Remark 2.1. As an aside, note that there is a special type of fundamental normal (or almost normal) surface called a *vertex fundamental* surface. These lie on extremal rays for a certain linear cone of embedded normal (almost normal) surfaces. They are thus vertices of the projectivization of the cone.

It is possible to certify that such surfaces are, in fact, vertex surfaces. This in turn implies that they are fundamental. Hass, Lagarias, and Pippenger [9] use this fact to certify connectedness of a normal surface and thus to show that unknotting is in **NP**. This is markedly different from general fundamental surfaces where it is not currently known how to certify that they are fundamental.

S. Ivanov has raised the possibility of using vertex fundamental two-spheres in order to certify that the S_i are connected. We have preferred to use Theorem 5.3 which relies on [1]. This is because we use [1] in an even more substantial way in Theorem 11.1. I would not need Theorem 11.1 if I knew how to certify that a triangulation is zero-efficient.

3. Definitions

We begin with a naive discussion of complexity theory. Please consult [7] for a more through treatment.

A *problem* P is a function from a set of finite binary strings, the *instances*, to another set of finite binary strings, the *answers*. A problem P is a *decision problem* if the range of P is the set $\{0, 1\}$. The length of a binary string in the domain of P is the *size* of the instance. A *solution* for P is a Turing machine $\mathcal{M}$ which, given input T in its tape, computes $P(T)$ and halts with only that output on its tape. We will engage in the usual abuse of calling such a Turing machine an *algorithm* or a *procedure*.

An algorithm is *polynomial time* if there is a polynomial $q\colon \mathbb{R} \to \mathbb{R}$ so that the Turing machine $\mathcal{M}$ finishes computing in time at most $q(\text{size}(T))$. Computing bounds on q, or even its degree, is often a difficult problem. We will follow previous treatments in algorithmic topology and leave this problem aside.

A decision problem P lies in the complexity class **NP** if there is a polynomial $q'\colon \mathbb{R} \to \mathbb{R}$ with the following property: For all instances T with $P(T) = 1$ there is proof of length at most $q'(\text{size}(T))$ that $P(T) = 1$. Such a polynomial length proof is a *certificate* for T. More concretely: Suppose that there is a polynomial q'' and a Turing machine $\mathcal{M}''$ so that, for every instance T with $P(T) = 1$, there is a string C where $\mathcal{M}''$ run on (T, C) outputs the desired proof that $P(T) = 1$ in time less than $q''(\text{size}(T))$. Then, again, the problem P is in **NP** and we also call C a certificate for T.

We now turn to topological considerations. A *model tetrahedron* τ is a copy of the regular Euclidean tetrahedron of side length 1 with vertices labelled by 0, 1, 2, and 3. See Figure 1 for a picture. Label the six edges by their vertices $(0, 1)$, $(0, 2)$, *etc.* Label the four faces by the number of the vertex they do *not* contain. The standard orientation on $\mathbb{R}^3$ induces an orientation on the model tetrahedron which in turn induces orientations on the faces.

A *labelled triangulation* of size n is a collection of n model tetrahedra $\{\tau_i\}_{i=1}^n$, each with a unique name, and a collection of *face pairings*. Here a face pairing is a triple (i, j, σ) specifying a pair of tetrahedra τ_i and τ_j as well as an orientation reversing isometry σ from a face of τ_i to a face of τ_j. We will omit the labellings when they are clear from context.

A triangulation is not required to be a simplical complex. However every face must appear in exactly two face pairings or in none. Also, no face may be glued to itself. We do not require for a face pairing (i, j, σ) that $i \neq j$.

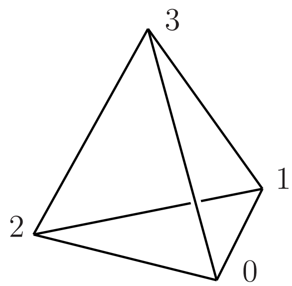

Figure 1. A regular Euclidean tetrahedron with all side-lengths equal to one.

Let $|T|$ be the *underlying topological space*; the space obtained by taking the disjoint union of the model tetrahedra and taking the quotient by the face pairings.

At this point we should fix an encoding scheme which translates triangulations into binary strings. However we will not bother to do more than remark that there are naive schemes which require about $n \log(n)$ bits to specify a triangulation with n tetrahedra. (This blow-up in length is due to the necessity of giving the tetrahedra unique names.) Thus it is only a slight abuse of language to say that a triangulation T has size n when in fact its representation as a binary string is somewhat longer.

Recall that the *three-sphere* is the three-manifold:

$$S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}.$$

The connect sum $M \# N$ of two connected three manifolds M and N is obtained by removing an open three-ball from the interior of each of M and N and gluing the resulting two-sphere boundary components. The connect sum naturally extends to a collection of connected three-manifolds; if M is the disjoint union of connected three-manifolds then $\# M$ denotes their connect sum.

Note that by Alexander's Theorem (Theorem 1.1 of [10]) $M \# S^3$ is homeomorphic to M, for any three-manifold M. So we adopt the convention that the empty connect sum yields the three-sphere.

We now give a slightly non-standard definition of *compression body*. Take S a closed orientable surface. Let $C_0 = S \times [0, 1]$. Choose a disjoint collection of simple closed curves in some component of $S \times \{0\}$ and attach two-handles in the usual fashion along these curves. Cap off some (but not necessarily all) of any resulting two-sphere boundary components with three-handles. The final result, C, is a *compression*

body. Set $\partial_+ C = S \times \{1\}$ and set $\partial_- C = \partial C \setminus \partial_+ C$. Our definition differs from others (*e.g.*, [4]) in that two-sphere components in $\partial_- C$ are allowed. The reasons for this are explained in Remark 10.2.

4. Preliminaries

Here we give a few algorithms which take triangulations and check topological properties. See [9] for a more in-depth discussion of these.

Theorem 4.1. *There is a polynomial time algorithm which, given a triangulation T, decides whether or not $|T|$ is a three-manifold.*

Proof. Apply Poincare's Theorem: Suppose the frontier of a regular neighborhood for each vertex of the triangulation is either a two-disk or a two-sphere. Then $|T|$ is a three-manifold. The converse also holds.

There are only linearly many (in size(T)) vertices. Checking that each frontier is a two-sphere or a disk takes time at most polynomial (again, in size(T)) as each is a union of at most linearly many normal triangles (see Section 5). Thus checking the hypothesis of Poincare's Theorem takes time at most polynomial in size(T). □

Recall that a three-manifold M is a *homology three-sphere* if it has the same homology groups as S^3.

Theorem 4.2. *There is a polynomial time algorithm which, given a triangulation T of a three-manifold, decides whether or not $|T|$ is a homology three-sphere.*

Proof. First apply Theorem 4.1 to check that $|T|$ is in fact a three-manifold. The homology groups $H_*(|T|, \mathbb{Z})$ may be read off from the Smith Normal Form of the chain boundary maps (see [5], Section 2, for an accessible overview of algorithmic computation of homology). Smith Normal Form of an integer matrix may be computed in polynomial time (see [11]). □

We also record, for future use, a few consequences of the homology three-sphere assumption:

Lemma 4.3. *If M^3 is a homology three-sphere then M is connected, closed, and orientable. Also every closed, embedded surface in M is orientable and separating. Finally, every connect summand of M is also a homology three-sphere.*

In particular no lens space (other than S^3) appears as a summand of a homology three-sphere.

We end this section with the simple:

Theorem 4.4. *There is a polynomial time algorithm which, given triangulations T and T', decides whether or not T is identical to T'.*

Proof. Recall that T and T' are labelled: all of the tetrahedra come equipped with names. To check for isomorphism simply check that every name appearing in T also appears in T' and that all of the face pairings in T and T' agree. □

Remark 4.5. Note that the labelling is not needed to determine isomorphism of triangulations. This is because an isomorphism is completely determined by the image of a single tetrahedron.

5. Normal and almost normal surfaces

In order to study triangulations we first discuss Haken's theory of *normal surfaces*. See [9] for a more through treatment.

On a face f of the model tetrahedron τ there are three kinds of properly embedded arc with end points in distinct edges of f. These are called *normal arcs*. A simple close curve $\alpha \subset \partial\tau$ is a *normal curve* if α is transverse to the one-skeleton of τ and α is a union of normal arcs. The *length* of a normal curve α is the number of normal arcs it contains. A normal curve α is called *short* if it has length three or four.

Lemma 5.1. *A normal curve $\alpha \subset \partial\tau$ misses some edge of τ^1 if and only if α meets every edge at most once if and only if α is short*

To see this let $\{v_{(i,j)} \mid 0 \leq i < j \leq 3\}$ be the number of intersections of α with each of the six edges of τ. There are twelve inequalities $v_{(0,1)} \leq v_{(1,2)} + v_{(0,2)}$, *etc*, as well as six equalities $v_{(0,1)} + v_{(1,2)} + v_{(0,2)} = 0$ mod 2, *etc*. Easy calculation gives the desired result.

In a model tetrahedron there are seven types of *normal disk*, corresponding to the seven distinct short normal curves in $\partial\tau$. See Figure 2. These are the four *normal triangles* and three *normal quads*. We have triangles of type 0, 1, 2, or 3 depending on which vertex they cut off of the model tetrahedron, τ. We have quads of type 1, 2, or 3 depending on which vertex is grouped with 0 when we cut τ along the quad.

A surface S properly embedded in $|T|$ is *normal* if $S \cap \tau$ is a collection of normal disks for every tetrahedron $\tau \in T$.

There is also the almost normal octagon and almost normal annulus, defined by Rubinstein [16]. See Figure 3 for examples. An octagon is a disk in the model tetrahedron bounded by a normal curve of length eight. An annulus is obtained by taking two disjoint normal disks and tubing them together along an arc parallel to an edge of the model tetrahedron. A surface S properly embedded in $|T|$ is *almost normal* if $S \cap \tau$ is a collection of normal disks, for every tetrahedron $\tau \in T$, except

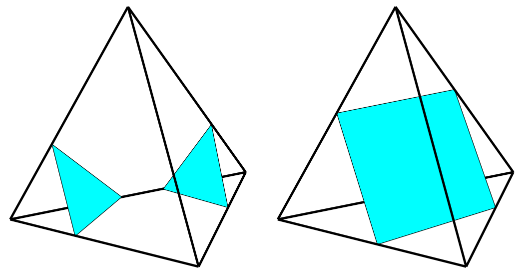

FIGURE 2. Two of the four triangles and one of the three quads.

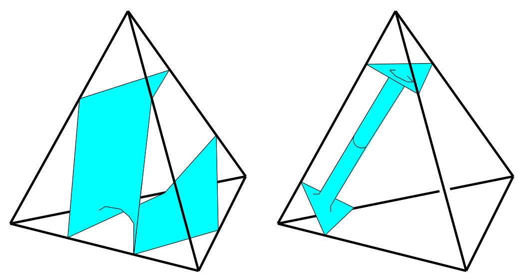

FIGURE 3. One of the three octagons and one of the 25 annuli.

one. In the exceptional tetrahedron there is a collection of normal disks and exactly one almost normal piece.

5.1. **Weight and Euler characteristic.** For an either normal or almost normal surface S take the *weight* of S, $\mathrm{weight}(S) = |S \cap T^1|$, to be the number of intersections between S and the one-skeleton T^1. Note that $\mathrm{size}(S)$, the number of bits required to describe S, is about $\mathrm{size}(T)\log(\mathrm{weight}(S))$. To see this record a normal surface S as a *surface vector* $v(S) \in \mathbb{Z}^{7 \cdot \mathrm{size}(T)}$ where the first $4 \cdot \mathrm{size}(T)$ coordinates describe the number of normal triangles of each type while the last $3 \cdot \mathrm{size}(T)$ coordinates describe the number of normal quads of each type. At least two-thirds of these last $3 \cdot \mathrm{size}(T)$ coordinates are zero, as an embedded surface has only one kind of normal quad in each tetrahedron.

For an almost normal surface S we again record the vector $v(S)$ of numbers of normal disks, as well as the type of the almost normal piece and the name of the tetrahedron containing it.

If two normal (or almost normal) surfaces S and S' have the same vector then S is *normally isotopic* to S'. This is the natural equivalence relation on these surfaces. As such we refer to normal or almost normal

surfaces and their vectors interchangeably where this does not cause confusion.

We now have a few results concerning normal and almost normal surfaces. We assume throughout that the triangulation T has underlying space a three-manifold.

Lemma 5.2. *There is an polynomial time algorithm which, given a triangulation T and a normal or almost surface vector $v(S)$, computes the weight of S and the Euler characteristic of S.*

Proof. To find the weight of S on a single edge e of T^1 count the number of normal disks meeting e (with multiplicity depending on how many times the containing tetrahedron meets e) and divide by the valency of e in T^2, the two-skeleton.

For the Euler characteristic simply use the formula $\chi(S) = F - E + V$ and the cell structure on S coming from its being a normal surface. (If S contains an almost normal annulus then we must add a single edge running between the two boundary components of the annulus.) Counting the number of faces and edges is straight-forward. The number of vertices equals the weight.

See [1], the end of Section 5, for a more detailed discussion. □

Theorem 5.3 (Agol, Hass Thurston [1])**.** *There is an polynomial time algorithm which, given a triangulation T and a normal or almost normal surface vector $v(S)$, produces the surface vectors for the connected components of S.*

A caveat is required here – if several normally isotopic copies of F appear in S then the algorithm of Theorem 5.3 produces $v(F)$ only once and also reports the number of copies. This is required if the algorithm is to run in polynomial time on input of the form $n \cdot F + G$.

Proof of Theorem 5.3. This is one application of the "extended counting algorithm" given in [1]. See the proof of Corollary 17 of that paper. □

5.2. **Vertex linking.** Fix a triangulation T of some three-manifold. Suppose $x \in |T|$ is a vertex of the triangulation. Let S be the frontier of small regular neighborhood of x. Then S is a connected normal surface which contains no normal quads. Such a surface is a *vertex linking* or simply a *vertex link*. A normal sphere which contains a normal quad is called *non-trivial*. If the triangulation contains no non-trivial normal two-spheres then T is *zero-efficient*.

5.3. **Haken sum.** Suppose S, F, G are three non-empty normal surfaces with $v(S) = v(F) + v(G)$. Then S is the *Haken sum* of F and G or, equivalently, S *decomposes* as a Haken sum. Likewise, suppose S and F are almost normal with identical almost normal piece, G is normal, and again $v(S) = v(F)+v(G)$. Then, again, S is a Haken sum. In either case we write $S = F + G$. If S, normal or almost normal, does not decompose as a Haken sum then S is *fundamental*. As an easy exercise:

Lemma 5.4. *If $S = F + G$, where G is a vertex link, then S is not connected.* □

Also,

Lemma 5.5. *If $S \subset |T|$ is a fundamental normal or almost normal surface then the largest entry of $v(S)$ is at most $\exp(\mathrm{size}(T))$.*

That is, there is a constant c (not depending on T or S) such that the largest entry is less than $2^{c \cdot \mathrm{size}(T)}$. This lemma is proved for normal surfaces by Lemma 6.1 of [9] and their proof is essentially unchanged for almost normal surfaces. □

Lemma 5.6. *Suppose T is a triangulation of a homology three-sphere. Suppose T contains a non-trivial normal two-sphere. Then T contains a non-trivial normal two-sphere which is fundamental.*

Proof. This is discussed as Proposition 4.7 of [13]. The essential points are that Euler characteristic is additive under Haken sum, that T does not contain any normal $\mathbb{RP}^2$ or $\mathbb{D}^2$ (by Lemma 4.3), and that no summand is vertex-linking (by Lemma 5.4). □

Lemma 5.7. *Suppose T is zero-efficient triangulation of a homology three-sphere. Suppose T contains an almost normal two-sphere. Then T contains a fundamental almost normal two-sphere.*

Proof. This is identical to the proof of Lemma 5.6, except that S cannot have any normal two-sphere summand as T is zero-efficient. □

Of a much different level of difficulty is Thompson's Theorem:

Theorem 5.8 (Thompson [18])**.** *Suppose $|T| \cong S^3$. Suppose also that T is zero-efficient. Then T contains an almost normal two-sphere.* □

We cannot do better than refer the reader to Thompson's original paper and remark that the proof uses Gabai's notion of *thin position* for knots [6].

We end this section with:

Lemma 5.9. *There is an exponential time algorithm which, given a triangulation T of a three-manifold, produces either the surface vector of a fundamental non-trivial normal two-sphere or, if none exists, produces the surface vector of a fundamental almost normal two-sphere or, if neither exists, reports "$|T|$ is not homeomorphic to the three-sphere".*

We only sketch a proof – the interested reader should consult [9], [13] (page 53), or [2] (page 83). If T is not zero-efficient there is a fundamental normal two-sphere. This can be found by enumerating all fundamental surfaces (a finite list, by work of Haken) and checking each surface on the list. If T is zero-efficient then no non-trivial normal two-sphere appears. However we again have that some fundamental almost normal two-sphere exists, if T contains any almost normal two-sphere. Finally, if no non-trivial normal sphere nor any almost normal sphere appears in amongst the fundamentals then, combining Lemma 5.7 and Theorem 5.8, conclude that $|T|$ is not the three-sphere.

As presented the running time of the algorithm is unclear; it depends on the number of fundamental surfaces. However, using vertex fundamental surfaces (see Remark 2.1) and linear programming techniques Casson reduces the search to take time at most a polynomial times $3^{\text{size}(T)}$. □

6. Blocked submanifolds

Normal (and almost normal) surfaces cut a triangulated manifold into pieces. These submanifolds have natural polyhedral structures which we now investigate.

Let τ be a model tetrahedron, and suppose that $S \subset \tau$ is a embedded collection of normal disks and at most one almost normal piece. Let B be the closure of any component of $\tau \smallsetminus S$. We call B a *block*. See Figure 4.

An block containing exactly two normal disks of the same type is called a *product block*. All other blocks are called *core blocks*. Note that there are only seven kinds of product block possible, corresponding to the seven types of normal disks. Likewise there is a bounded number of core blocks. Five such are shown in Figure 4, but many more are possible. Most of these meet an almost normal annulus.

The components of ∂B meeting S are the *horizontal* boundary components of B, denoted $\partial_h B$. All other faces of B (the faces of B which lie in the two-skeleton) are $\partial_v B$, the *vertical* boundary.

Suppose now that T is a triangulation of a three-manifold and $S \subset |T|$ is a normal or almost normal surface. For simplicity, suppose that

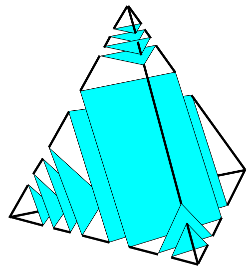

FIGURE 4. The tetrahedron τ cut along S. Note that there are two blocks of the form "normal disk cross interval".

S is a transversely oriented and separating. Let N_S be the closure of the component of $|T| \smallsetminus S$ into which the transverse orientation points.

Then N_S is a *blocked submanifold* of $|T|$. Note that the induced cell structure on N_S (coming from T) naturally breaks into two parts. So, let $\widehat{N}_P$ be the union of all product blocks in N_S and let $\widehat{N}_C$ be the union of all core blocks in N_S.

Remark 6.1. In any blocked submanifold there are at most a linear number (in $\text{size}(T)$) of core blocks. In fact there at most six in each tetrahedron (plus possibly two more coming from the almost normal annulus). This simple observation underlies Kneser-Haken finiteness (see, for example, [8]) and is generally useful in the algorithmic setting.

Note that $\widehat{N}_P$ and $\widehat{N}_C$ are not necessarily submanifolds of $|T|$. To produce submanifolds take N_P to be a regular closed neighborhood of $\widehat{N}_P$, where the neighborhood is taken inside of N_S. Also, take N_C to be the closure of $N_S \smallsetminus N_P$. Note the asymmetry between the definitions of N_P and N_C: we have $\widehat{N}_P \subset N_P$ while $N_C \subset \widehat{N}_C$. As above define $\partial_h N_P = N_P \cap S$ and $\partial_v N_P = \overline{\partial N_P \smallsetminus \partial_h N_P}$. The horizontal and vertical boundaries $\partial_h N_C$ and $\partial_v N_C$ are defined similarly.

Note also that $\widehat{N}_P$ may be represented by a *block vector*; an element $v(\widehat{N}_P) \in \mathbb{Z}^{7 \cdot \text{size}(T)}$ where the first $4 \cdot \text{size}(T)$ coordinates describe the number of triangle product blocks of each type while the last $3 \cdot \text{size}(T)$ coordinates describe the number of quad product blocks of each type.

We now have:

Theorem 6.2. *There is an polynomial time algorithm which, given a triangulation T and blocked submanifold $N_S \subset |T|$ (via the surface vector $v(S)$), produces the block vector for each connected component of $\widehat{N}_P$.*

We only require this algorithm for blocked submanifolds N_S cut out of $|T|$ by a transversely oriented separating normal or almost normal two-sphere S. In this case, the input to the algorithm is just T and the surface vector $v(S)$. From $v(S)$ it is possible (in time polynomial in size(T) and log(weight(S))) to find a block vector for $\widehat{N}_P$. Given this the other details for a proof of Theorem 6.2 can be found in [1]. $\square$

Remark 6.3. We also remark that $\widehat{N}_P$ has at most a linear number (in size(T)) of connected components. This is because $\partial_v N_P = \partial_v N_C$ and the latter has at most linearly many components. (See Remark 6.1.) Thus, unlike the algorithm of Theorem 5.3 we do not need to concern ourselves with components of $\widehat{N}_P$ appearing with high multiplicity.

7. Normalizing slowly

In this section we discuss a restricted version of Haken's *normalization* procedure for producing normal surfaces. This material appeared first in an unpublished preprint of mine and later in my thesis [17]. I thank Danny Calegari for reading an early version of this work. I also thank Bus Jaco for several enlightening conversations regarding Corollary 9.3.

Several authors have independently produced versions of these ideas; for example see [13], [2], or [15].

Let $S \subset |T|$ be a transversely oriented, almost normal surface. Here T is triangulation of a closed, orientable, connected three-manifold.

Definition. A compression body $C_S \subset |T|$ is *canonical* for S if $\partial_+ C_S = S$, $\partial_- C_S$ is normal, the transverse orientation points into C_S, and any normal surface S' disjoint from S may be normally isotoped to one disjoint from C_S.

As a bit of notation take $\partial_- C_S = \widetilde{S}$ and call this the *normalization* of S.

Theorem 11.1. *Given a transversely oriented almost normal surface S there exists a canonical compression body C_S and C_S is unique (up to normal isotopy). Furthermore there is a algorithm which, given the triangulation T and the surface vector $v(S)$, computes the surface vector of $\partial_- C_S = \widetilde{S}$.*

The proof of this theorem is lengthy and is accordingly spread from Section 8 to Section 11. We here give the necessary definitions. In Section 8 we discuss the tightening procedure. In Section 9 we show that the tightening procedure gives an embedded isotopy. We discuss

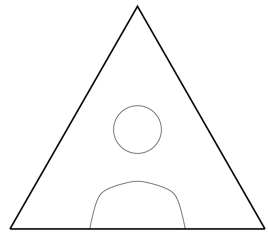

FIGURE 5. A simple curve and a bent arc.

the capping off procedure in Section 10. Finally in Section 11 we prove Theorem 11.1.

7.1. **Non-normal surfaces.** Let S be a surface properly embedded in a triangulated three-manifold $|T|$ and suppose that S is transverse to the skeleta of T. Denote the i-skeleton of T by T^i. Generalize the notion of weight so that $\text{weight}(S) = |S \cap T^1|$.

We characterize some of the ways S can fail to be normal. A *simple curve* of S is a simple closed curve of intersection between S and the interior of some triangular face $f \in T^2$. Also, a *bent arc* of S is a properly embedded arc of intersection between S and the interior of some triangular face $f \in T^2$ with both endpoints of the arc contained in a single edge of f. Both of these are drawn in Figure 5.

7.2. **Surgery and tightening disks.**

Definition. An embedded disk $D \subset |T|$ is a *surgery disk* for S if $D \cap S = \partial D$, $D \subset T^2$ or $D \cap T^2 = \emptyset$, and $D \cap T^1 = \emptyset$.

There is a *surgery* of S along D: Remove a small neighborhood of ∂D from S and cap off the boundaries thus created with disjoint, parallel copies of D. Note that we do not require ∂D to be essential in S. A simple curve of $S \cap T^2$ is *innermost* if it is the boundary of a surgery disk embedded in a triangle of T^2.

Definition. An embedded disk $D \subset |T|$ is a *tightening disk* for S if $\partial D = \alpha \cup \beta$ where $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $D \cap S = \alpha$. Also, either $D \subset T^2$ or $D \cap T^2 = \beta$. But in either case $D \cap T^1 = \beta$, and $D \cap T^0 = \emptyset$.

There is a *tightening isotopy* of S across D: Push α along the disk D, via ambient isotopy of S supported in a small neighborhood of D, until α moves past β. This procedure reduces $\text{weight}(S)$ by exactly two. A bent arc of S is *outermost* if it lies on the boundary of a tightening disk embedded in a triangle of T^2.

Suppose S contains an almost normal octagon. There are two tightening disks on opposite sides of the octagon both giving tightening isotopies of S to a possibly non-normal surface of lesser weight. (See Figure 3.) We call these the *exceptional tightening disks*. If S contains an almost normal annulus then the tube is parallel to at least one edge of the containing tetrahedron. (Again see Figure 3.) For every such edge there is an exceptional tightening disk. Also, the disk which surgers the almost normal annulus will be called the *exceptional surgery disk*.

7.3. **Normal isotopy.** We sharpen our notion of equivalence:

Definition. An isotopy $H : |T| \times I \to |T|$ is a *normal isotopy* if, for all $s \in I$ and for every simplex σ in T, $H_s(\sigma) = \sigma$.

Two subsets of $|T|$ are *normally isotopic* if there is a normal isotopy taking one to the other.

8. Tightening

This section discusses the *tightening* procedure which will yield an embedded isotopy. This is proved in Lemma 9.1 below.

Suppose that $S \subset |T|$ is a transversely orientable separating almost normal surface. Here T is a triangulation of a three-manifold. We wish to isotope S off of itself while continuously reducing the weight of S.

Suppose that D is an exceptional tightening disk for S. Choose the transverse orientation for S which points into the component of $|T| \smallsetminus S$ which meets D. The *F-tightening procedure* constructs a map $\mathcal{F}\colon S \times [0, n] \to |T|$ as follows:

(1) Let $F_0 = S$. Take $\mathcal{F}_0\colon S \times \{0\} \to |T|$ to be projection to the first factor. Let $D_0 = D$.
(2) Now do a small normal isotopy of F_0 in the transverse direction while tightening F_0 along D_0. This extends $\mathcal{F}_0$ to a map $\mathcal{F}_1\colon S \times [0, 1] \to |T|$, with $F_t = \mathcal{F}_1(S \times \{t\})$. Note that the surface F_1 inherits a transverse orientation from F_0. Arrange matters so that $F_{\frac{1}{2}}$ is the only level which is not transverse to T^2. Furthermore $F_{\frac{1}{2}}$ only has a single tangency with T^1 and this tangency occurs in the middle of $\partial D_0 \cap T^1$.
(3) At stage $k \geq 1$ there are two possibilities. Suppose first that F_k has an outermost bent arc α with the transverse orientation of F_k pointing into the tightening disk D_k, which is cut out of T^2 by α. Then extend $\mathcal{F}_k$ to $\mathcal{F}_{k+1}\colon S \times [0, k+1] \to |T|$ by doing a small normal isotopy of F_k in the transverse direction

while tightening F_k across D_k, the k^{th} tightening disk. So $\mathcal{F}_k = \mathcal{F}_{k+1}|S\times[0,k]$ and $F_t = \mathcal{F}_{k+1}(S\times\{t\})$. Note that the surface F_{k+1} inherits a transverse orientation from F_k. Arrange matters so that $F_{k+\frac{1}{2}}$ is the $k+1^{\text{th}}$ level which is not transverse to T^2. Furthermore $F_{k+\frac{1}{2}}$ only has a single tangency with T^1 and this tangency occurs in the middle of $\partial D_k \cap T^1$.

If there is no outermost bent arc $\alpha \subset F_k$ then set $n = k$ and the procedure halts.

Remark 8.1. As $\text{weight}(F_{k+1}) = \text{weight}(F_k) - 2$ this process terminates. Note also that $\mathcal{F}_n$ is far from unique – at any stage in the procedure there may be many tightening disks to choose from.

We will show in Lemma 9.1 that the map $\mathcal{F}_n \colon S\times[0,n] \to M$ is an embedding. Note that, by construction, $S = F_0 = \mathcal{F}_n(S\times\{0\})$ and in general $F_t = \mathcal{F}_n(S\times\{t\})$. To simplify notation set $\mathcal{F} = \mathcal{F}_n$.

9. Tracking the isotopy

In this section we analyze how $\text{image}(\mathcal{F})$ intersects the skeleta of the triangulation. Let $S \subset |T|$, $\mathcal{F}$, $\mathcal{F}_k$, and F_t be as defined in Section 8.

Figures 6 and 7 display a few of the possible components of intersection $f \cap \text{image}(\mathcal{F}_k)$ assuming that $\mathcal{F}_k$ is an embedding. Here f is a face of T^2. Lemma 9.1 below shows that this collection is complete up to symmetry. Note that the normal arcs, bent arcs, and simple curves bounding the components receive a transverse orientation from S or F_k. In these figures all arcs of S are pointed towards while arcs of F_k are pointed away from, agreeing with the transverse orientation. The components of intersection containing a normal arc of F_k are called *critical*. Those with a bent arc of F_k are called *temporary*. Any component containing a simple curve of F_k is called *terminal with hole*. Finally, components of $f \cap \text{image}(\mathcal{F}_k)$ which are completely disjoint from F_k are simply called *terminal*. Again, refer to Figure 6 and 7.

The tightening procedure combines the critical components in various ways. However, a temporary component always results in a terminal (possibly with hole) and these are stable. Note also that there is a second critical rectangle which "points upward." The non-critical components may be foliated by the levels of $\mathcal{F}_k$ in multiple ways, depending on the order of the tightening isotopies.

Lemma 9.1. *For every k, the map $\mathcal{F}_k$ is an embedding. Furthermore, for $k > 0$ and for every $f \in T^2$, the connected components of $f \cap \text{image}(\mathcal{F}_k)$ are given, up to symmetry, by Figures 6 and 7.*

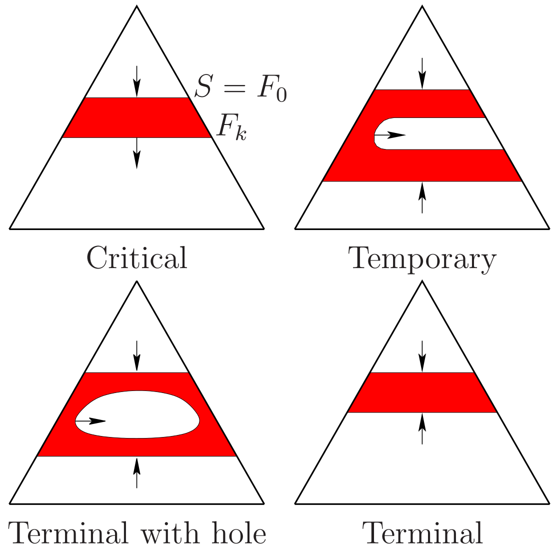

FIGURE 6. The Rectangles

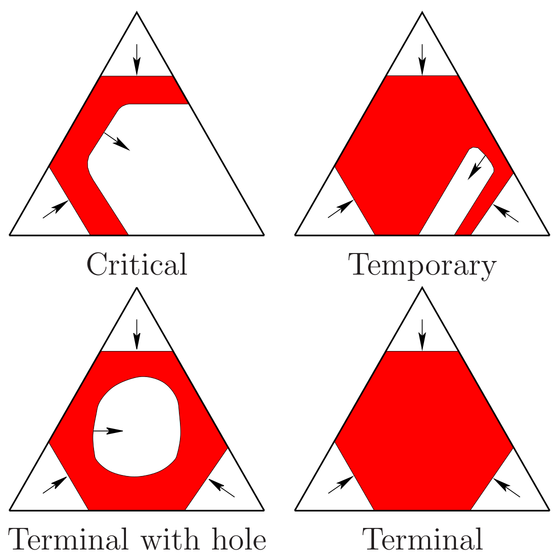

FIGURE 7. The Hexagons

Proof. Proceed by induction: Both claims are trivial for $k = 0$. Now to deal with $k = 1$. The exceptional tightening disk D_0 has interior disjoint from $S = F_0$. It follows that $\mathcal{F}_1$ is an embedding. To verify the second claim for $k = 1$ note that the image of $\mathcal{F}_1|S \times [0, \epsilon]$ intersects all faces $f \in T^2$ only in critical rectangles. Up to $t = \frac{1}{2}$ the image

of $\mathcal{F}_1|S\times[0,t]$ intersected with f is combinatorially constant. Crossing $t = \frac{1}{2}$ adds a regular neighborhood of D_0 to the image. This only intersects f in a regular neighborhood of $\partial D_0 \cap T^1$. So the pieces of $f \cap \text{image}(\mathcal{F}_1)$ are unions of critical rectangles connected by small neighborhoods of sub-arcs of T^1. Also these sub-arcs only meet the F_t side of the critical rectangles. As each critical rectangle meets two edges of the face f it follows that at most three critical rectangles are joined together to form a component of $f \cap \text{image}(\mathcal{F}_1)$. We list all possible cases – consulting Figures 6 and 7 will be helpful:

(1) Two critical rectangles in f may be combined to produce a temporary rectangle, a terminal rectangle with a hole, or a critical hexagon.
(2) Three critical rectangles in f may be combined to produce a temporary hexagon or a terminal hexagon with a hole.

Now to deal with the general case: Suppose that both claims hold at stage k. Suppose that $\alpha \subset F_k$ is the bent arc on the boundary of $D_k \subset f \in T^2$, the next tightening disk in the sequence. Suppose that interior(D_k) meets image$(\mathcal{F}_k)$. By the second induction hypothesis there is a component, C, of $f \cap \text{image}(\mathcal{F}_k)$ which meets interior(D_k) and appears among those listed in Figures 6 and 7. Observe that each component of $f \cap \text{image}(\mathcal{F}_k)$, and hence C, meets at least two edges of f. The bent arc α meets only one edge of f. It follows that the interior of C must meet α. Thus $\mathcal{F}_k$ was not an embedding, a contradiction.

It follows that $D_k \cap \text{image}(\mathcal{F}_k) = \alpha$. Since the $k+1^{\text{th}}$ stage of the isotopy is supported in a small neighborhood of $F_k \cup D_k$ it follows that $\mathcal{F}_{k+1}$ is an embedding.

Now, the transverse orientation on F_k gives rise to a transverse orientation on F_{k+1}. To verify the second claim again list the possible cases:

(1) Two critical rectangles in f may be combined to produce a temporary rectangle, a terminal rectangle with a hole, or a critical hexagon.
(2) Three critical rectangles f may be combined to produce a temporary hexagon or a terminal hexagon with a hole.
(3) A critical rectangle and critical hexagon in f may be combined to produce a temporary hexagon or a terminal hexagon with a hole.
(4) A temporary component can lead only to a terminal (possibly with hole).

This completes the induction. □

Remark 9.2. By maximality of $\mathcal{F}$, the surface $F_n = \mathcal{F}(S \times \{n\})$ has no outermost bent arcs with outward orientation. A bent arc with inward orientation would violate the second induction hypothesis of Lemma 9.1. So F_n contains no bent arcs. F_n may contain simple curves, but the second induction hypotheses shows that all of these are innermost with transverse orientation pointing toward the bounded surgery disk.

Given that $\mathcal{F}$ is an embedding, in the sequel image($\mathcal{F}_k$) is denoted by $\mathcal{F}_k$. Replacing S in Lemma 9.1 by a disjoint union of S with a collection of normal surfaces gives:

Corollary 9.3. *If S' is any normal surface in $|T|$ which does not intersect S then $\mathcal{F} \cap S' = \emptyset$, perhaps after a normal isotopy of S' (rel S).*

Let τ be any tetrahedron in the given triangulation T.

Lemma 9.4. *For all $k \geq 1$, $\tau \smallsetminus \mathcal{F}_k$ is a disjoint collection of balls.*

Proof. Again we use induction. Our induction hypothesis is as follows: $\tau \smallsetminus \mathcal{F}_k$ is a disjoint collection of balls, unless $k = 0$, and τ contains the almost normal annulus of S. (In this situation $\tau \smallsetminus \mathcal{F}_0$ is a disjoint collection of balls and one solid torus $\mathbb{D}^2 \times S^1$.)

The base case is trivial. Suppose B is a component of $\tau \smallsetminus \mathcal{F}_k$. There are now two cases to consider. Either B is cut by an exceptional tightening disk or it is not. Assume the latter. Then B is a three-ball by induction and after the $k+1^{\text{th}}$ stage of the isotopy $B \cap \mathcal{F}_{k+1}$ is a regular neighborhood (in B) of a collection of disjoint arcs and disks in ∂B. Hence $B \smallsetminus \mathcal{F}_{k+1}$ is still a ball.

If B is adjacent to the almost normal piece of F_0 then let D_0 be the exceptional tightening disk. Set $B_\epsilon = B \smallsetminus \eta(D_0)$. Each component of B_ϵ is a ball, and the argument of the above paragraph shows that they persist in the complement of $\mathcal{F}_1$. □

A similar induction argument proves:

Lemma 9.5. *For all $k \geq 1$, $\tau \cap \mathcal{F}_k$ is a disjoint collection of handlebodies.*

This lemma is not used in what follows and its proof is accordingly left to the interested reader. Recall that $\partial \mathcal{F}_k = S \cup F_k$. A trivial corollary of Lemma 9.4 is:

Corollary 9.6. *For all k, the connected components of $\tau \cap F_k$ are planar.*

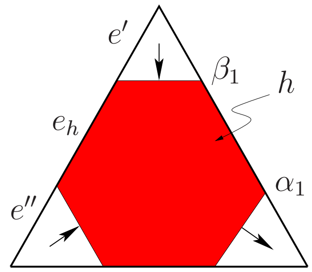

FIGURE 8. The normal arcs α_1 and β_1 are on the boundary of the critical hexagon h. Note that β does not meet e' or interior(h).

The connected components of $\tau \cap F_n$ warrant closer attention:

Lemma 9.7. *Each component of $\tau \cap F_n$ has at most one normal curve boundary component. This normal curve must be short.*

Proof. Let $\tau \in T$ be a tetrahedron. Let P be a connected component of $\tau \cap F_n$. By Lemma 9.1 the boundary ∂P is a collection of simple curves and normal curves in $\partial\tau$. Let α be any normal curve in ∂P. Let $\{\alpha_j\}$ be the normal arcs of α.

Claim. α has length three or four.

Call the collection of critical rectangles and hexagons (in $\partial\tau \cap \mathcal{F}$) meeting α the *support* of α. To prove the claim we have two cases. First suppose that only critical rectangles support α. So α is normally isotopic to a normal curve $\beta \subset \partial\tau \cap S$. The first step of the tightening procedure prevents β from being a boundary of the almost normal piece of S. It follows that α must be short.

Otherwise α_1, say, is on the boundary of a critical hexagon $h \subset f$. Let β be a normal curve of S meeting h and let $\beta_1 \subset \beta$ be one of the normal arcs in ∂h. Let e be the edge of f which α_1 does not meet. This edge is partitioned into three pieces; $e_h \subset h$, e', and e''. We may assume that β_1 separates e_h from e'. See Figure 8.

Note that a normal curve of length ≤ 8 has no parallel normal arcs in a single face. Thus β meets e' exactly once at an endpoint of e'. Since α and β do not cross it follows that β separates α from e' in $\partial\tau$.

Similarly, α is separated from e''. Thus α does not meet e at all. By Lemma 5.1 the normal curve α is short. This finishes the proof of the claim. $\square$

Claim. The component $P \subset \tau \cap F_n$ has at most one boundary component which is a normal curve.

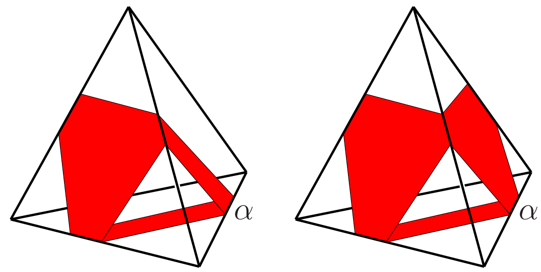

FIGURE 9. Diagrams for cases (1b) and (1c).

Proving this will complete the lemma. So suppose that ∂P contains two normal curves: α and β. Let A be the annulus cobounded by α and β in $\partial\tau$, the boundary of the model tetrahedron.

Suppose now that the transverse orientation F_n induces on α points away from A. Thus A and the support of α intersect. There are several cases to examine, depending on the length of α and the components of the support of α.

(1) Suppose α has length three:
  (a) If only critical rectangles support α then a normal triangle of S separates α and β.
  (b) If one critical hexagon and two critical rectangles support α then the almost normal octagon and the exceptional tightening disk together separate α and β. (See left hand side of Figure 9.)
  (c) If two critical hexagons and one critical rectangle support α then either a normal triangle or normal quad of S separates α and β. (See right hand side of Figure 9.)
  (d) If only critical hexagons support α then a normal triangle of S separates α and β.

(2) Suppose α has length four:
  (a) If only critical rectangles support α then a normal quad of S separates α and β.
  (b) If one critical hexagon and three critical rectangles support α then S could not have been an almost normal surface. (See left hand side of Figure 10.)
  (c) If two critical hexagons and two critical rectangles support α then a normal triangle of S separates α and β. (See right hand side of Figure 10.)

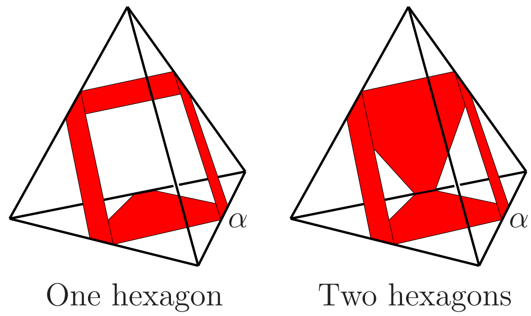

Figure 10. Diagrams for cases (2b) and (2c).

When α has length four it cannot be supported by more than two critical hexagons.

To recap: in all cases except 1(b) and 2(b), the support of α (possibly together with a terminal rectangle or hexagon) closes up, implying the existence of a normal disk of S with boundary a core curve of the annulus A. As this disk lies in S observe that $S \cap P \neq \emptyset$ and thus $S \cap F_n \neq \emptyset$. This contradicts the fact that $\mathcal{F}$ is an embedding (Lemma 9.1). Case 1(b) is similar, except that the support of α meets other critical or terminal components to form the octagon piece of S. So P must intersect either S or the exceptional tightening disk, again a contradiction of Lemma 9.1. Lastly, in case 2(b), S could not have been almost normal.

So deduce that the transverse orientation which F_n gives α must point toward A. Thus A and the support of α are disjoint. Let γ be an arc which runs along P from α to β. Let α' be a push-off of α along A, towards β. This push-off bounds a disk in one of the components of $\tau \smallsetminus \mathcal{F}$, by Lemma 9.4. This disk does not intersect $P \subset F_n \subset \mathcal{F}$ and hence fails to intersect γ. This is a contradiction. $\square$

Remark 9.8. By Lemma 9.1 all simple curves of F_i are innermost. It follows that the "tubes" analyzed in Lemma 9.7 do not run through each other. In addition, analysis similar to that needed for Lemma 9.5 implies that these tubes are unknotted, but this last fact is not needed in the sequel.

10. Capping off

Here we construct our candidate for C_S, the canonical compression body.

Let $\mathcal{F} \subset |T|$ be the image of the map constructed above. Recall that $\partial \mathcal{F} = S \cup F_n$ where S is the almost normal surface we started with

and F_n is the surface obtained by "tightening" S. Note that, since $\mathcal{F}$ is the embedded image of $S \times [0, n]$, in fact F_n is isotopic to S in $|T|$. (It cannot be normally isotopic as it has lower weight.)

Definition. A two-sphere which is embedded in $|T|$ but disjoint from T^2 is called a *bubble*.

We have a corollary which is easy to deduce from Lemma 9.7, Corollary 9.6, and Remark 9.8:

Corollary 10.1. *Let F'_n be the surface obtained by surgering all simple curves of F_n. Then F'_n is a disjoint collection of bubbles and normal surfaces. Each bubble bounds a ball with interior disjoint from $T^2 \cap F'_n$.*

Construct C_S as follows: For every simple curve α of F_n attach a two-handle to $\mathcal{F}$ along α. Attach so that the core of the two-handle is the subdisk of T^2 cut out by α. Call this $\mathcal{F}'$. As noted in Remark 9.8 all simple curves of F_n are innermost. So $\mathcal{F}'$ is an embedded compression body. At this point there may be components of $\partial_- \mathcal{F}'$ which are not normal. By Corollary 10.1 all of these are bubbles bounding a ball disjoint from all of the other bubbles. So cap off each bubble to obtain C_S. Set $\widetilde{S} = \partial_- C_S$. The next section proves that $v(\widetilde{S})$ does not depend on the choices made in the construction of $\mathcal{F}$.

Remark 10.2. The reason why two-spheres are allowed in $\partial_- C_S$ should now be clear: we cannot prevent normal two-spheres from appearing in the normalization process. In particular, if S is an almost normal two-sphere then, for one of the two possible transverse orientations, there will always be a normal two-sphere appearing in $\widetilde{S}$.

11. Proof of the normalization theorem

Suppose that S is almost normal and equipped with a transverse orientation. Before proving Theorem 11.1 recall that C_S, a compression body in $|T|$, is *canonical* for S if $\partial_+ C_S = S$, $\partial_- C_S$ is normal, the transverse orientation on S points into C_S, and any normal surface $S' \subset |T|$ may be normally isotoped to one disjoint from C_S.

We now have:

Theorem 11.1. *Given a transversely oriented almost normal surface S there exists a canonical compression body C_S and C_S is unique (up to normal isotopy). Furthermore there is a algorithm which, given the triangulation T and the surface vector $v(S)$, computes the surface vector of $\partial_- C_S = \widetilde{S}$.*

Proof. We proceed in several steps.

Claim. A canonical compression body C_S exists.

There are two cases. Either the transverse orientation for S points at the exceptional surgery disk (implying that S contained an almost normal annulus) or the transverse orientation points at an exceptional tightening disk.

In the first case, C_S is obtained by thickening S slightly and adding a regular neighborhood of the exceptional surgery disk. It is clear that C_S is a compression body, $\partial_+ C_S = S$, and $\partial_- C_S$ is normal. Suppose that S' is any normal surface in T which is disjoint from S. Then, perhaps after a normal isotopy of S' (rel S), we have that S' is disjoint from the exceptional surgery disk for S. It follows that S' may be isotoped out of C_S.

In the second case the transverse orientation of S points at an exceptional tightening disk of S. As in Section 8 form $\mathcal{F}$ with $\partial\mathcal{F} = S \cup F_n$. As in Section 10 attach two-handles to $\mathcal{F}$ along the simple curves of F_n to obtain $\mathcal{F}'$. Cap off the bubbles with their three-balls to obtain C_S. Again, C_S is a compression body with $\partial_+ C_S = S$.

Suppose now that S' is some normal surface in T which is disjoint from S. Then, by Corollary 9.3, the surface S' is disjoint from $\mathcal{F}$ (perhaps after a normal isotopy of S' rel S). Since S' is normal it cannot meet any of the disks (in T^2) bounded by simple curves of F_n. So $S' \cap \mathcal{F}' = \emptyset$ as well. Finally, suppose that A is a bubble component of $\partial_- \mathcal{F}'$. Let B be the three-ball which A bounds (such that $B \cap T^2 = \emptyset$). Then no component of S' meets B as $S' \cap A = \emptyset$ and S' is normal. Deduce that $S' \cap C_S = \emptyset$. The claim is complete.

Claim. The canonical compression body C_S is unique (up to normal isotopy).

Suppose that C_S and C'_S are both canonical compression bodies for S. Let $A = \partial_- C_S$ and $A' = \partial_- C'_S$. Then A and A' are normal surfaces, both disjoint from S. It follows that there exists a normal isotopy $\mathcal{H}$ which moves A' out of C_S, rel S, and conversely another normal isotopy $\mathcal{H}'$ which moves A out of C'_S, rel S.

Consider any face $f \in T^2$ and any normal arc $\alpha \subset f \cap S$. Let $X \subset f \cap C_S$ be the component containing α. Also take X' to be the component of $f \cap C'_S$ which contains α. We must show that X and X' have the same combinatorial type. Suppose not. After possibly interchanging X and X' there are only six situations to consider:

(1) X is a critical rectangle and X' is a terminal rectangle.
(2) X is a critical rectangle and X' is a critical hexagon.
(3) X is a critical rectangle and X' is a terminal hexagon.

(4) X is a critical hexagon and X' is a terminal hexagon.

In any of these four cases let δ be the normal arc of $A = \partial_- C_S$ on the boundary of X. Note that $\partial X'$ contains α (as does ∂X) and also another normal arc $\beta \subset f \cap S$ which does not meet X (as $S = \partial_+ C_S$). Now note that it is impossible for $\mathcal{H}'$ to normally isotope δ out of X' while keeping S fixed pointwise (as δ would have to cross β).

(5) X is a terminal rectangle and X' is a critical hexagon.

(6) X is a terminal rectangle and X' is a terminal hexagon.

In either of these cases let β be the other normal arc of $S \cap \partial X$. Then β intersects the interior of X', a contradiction.

This proves the claim.

Claim. There is a algorithm which, given the triangulation T and the surface vector $v(S)$, computes the surface vector of $\partial_- C_S = \widetilde{S}$.

We follow the proof of Lemma 9.1: We keep track of the intersection between the image of $\mathcal{F}_k$ and every face $f \in T^2$. These are unions of components, with all allowable kinds shown (up to symmetry) in Figures 6 and 7. There is at most one hexagon in each face and perhaps many rectangles, arranged in three families, one for each vertex of f. At stage n there are no bent arcs remaining. Now delete all simple curves of F_n and all normal arcs of S. The normal arcs left completely determine $\widetilde{S}$ and from this we may find the surface vector $v(\widetilde{S})$. This proves the claim and finishes the proof of Theorem 11.1. □

Of course, the algorithm just given is inefficient. It depends polynomially on size(T) and weight(S). In the next section we improve this to a algorithm which only depends polynomially on size(T) and log(weight(S)).

As a corollary of Theorem 11.1:

Corollary 11.2. *If $S \subset |T|$ is a transversely oriented almost normal two-sphere then C_S is a three-ball, possibly with some open three-balls removed from its interior. (These have closures disjoint from each other and from S.)*

Now an orientable surface in an orientable three-manifold may be transversely oriented in exactly two ways. By Theorem 11.1, if S is an almost normal surface, for each transverse orientation there is a canonical compression body. Call these C_S^+ and C_S^-.

From Corollary 11.2 deduce:

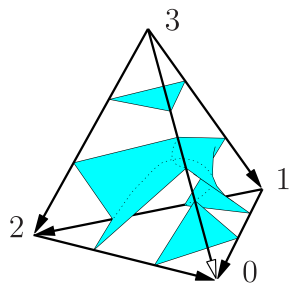

FIGURE 11. A one tetrahedron triangulation of S^3. It is a simple exercise to list all normal and almost normal surfaces in T. The reader may further amuse herself by drawing the graph T^1 as it actually sits in S^3. Slightly harder is to draw the two-skeleton.

Theorem 11.3. *If $S \subset |T|$ is an almost normal two-sphere and both $\partial C_S^+ \setminus S$ and $\partial C_S^- \setminus S$ are (possibly empty) collections of vertex-linking two-spheres, then $|T|$ is the three-sphere.*

Proof. By hypothesis $\partial C_S^+ \setminus S$ is a collection of vertex linking spheres. For each of these add to C_S^+ the corresponding vertex neighborhood. Let $\mathbb{B}^+$ be the resulting submanifold of $|T|$. By Alexander's Trick $\mathbb{B}^+$ is a three-ball. Do the same to C_S^- to produce $\mathbb{B}^-$. Applying Alexander's Trick again deduce that the manifold $|T| = \mathbb{B}^+ \cup_S \mathbb{B}^-$ is the three-sphere. □

12. An example

Here we give a brief example of the normalization procedure. Let T be the one vertex triangulation shown in Figure 11.

The front two faces (1 and 2) are glued to each other as are the back faces (0 and 3). The faces are glued to give the edge identifications shown. The surface S depicted in T is an almost normal two-sphere with two triangles and one almost normal octagon. It is easy to check this by computing $\chi(S) = 3 - 7 + 6 = 2$.

The sphere S has two exceptional tightening disks: D meeting the edge $(0, 3)$ of the model tetrahedron and D' meeting edge $(1, 2)$.

Tightening along D gives F_1 which is the vertex link. Tightening along D' gives F_1', F_2', F_3'. Here F_3' is a weightless two-sphere in T with a single simple curve and no other intersection with the two-skeleton. As a note of caution: F_1' drawn in the model tetrahedron has four bent arcs – however $F_1' \cap T^2$ contains only two. These are independent of each other and doing these moves in some order gives F_2' and F_3'. To

obtain the normalization of S on the D' side, surger the simple curve of F_3' and cap off the two resulting bubbles.

So, on the D side of S the normalization is the vertex link. On the D' side the normalization is the empty set. It follows from Theorem 11.3 that $|T|$ is the three-sphere. This finishes the example.

13. Normalizing quickly

The normalization procedure can be made much more efficient.

Theorem 13.1. *There is a polynomial time algorithm which, given T and the surface vector $v(S)$, produces as output $v(\widetilde{S})$, the normalization of S. Here S is assumed to be a separating transversely oriented almost normal surface and T is assumed to be a triangulation of a three-manifold.*

Remark 13.2. In fact S need not be separating. However the proof is somewhat simpler in the separating case and is all we require in what follows.

Recall that N_S is the closure of the component of $|T| \smallsetminus S$ which the transverse orientation points into. Then $\widehat{N}_P$ is the union of all product blocks in N_S and $\widehat{N}_C$ is the union of all the core blocks. Also N_P is a regular neighborhood of $\widehat{N}_P$, taken in N_S. Finally $N_C = \overline{N_S \smallsetminus N_P}$. We will prove Theorem 13.1 by altering our original normalization procedure three times. First we will show that the order of the tightening moves is irrelevant. Then we show that surgeries on simple curves and capping off of bubbles may happen during the normalization process, instead of being held until the end. Finally we show that tightening inside of N_P can be done very quickly. These three modifications combine to give an efficient algorithm.

13.1. **Changing the order of the tightening moves.** As stated in Remark 8.1 the isotopy $\mathcal{F}\colon S \times [0, n] \to M$ need not be unique. But we do have:

Lemma 13.3. *Any order for the tightening moves (performed in the construction of $\mathcal{F}$) gives the same surface $\widetilde{S}$ once the simple curves of F_n have been surgered.*

This follows immediately from the first sentence of Theorem 11.1.

13.2. **Surgery on simple curves and deleting bubbles.** We now alter the tightening procedure in a more substantial fashion:

Recall that $S \subset |T|$ is a transversely orientable separating almost normal surface. Recall that D is the exceptional tightening disk for

S. Transversely orient S to point into the component of $|T| \setminus S$ which meets D. Here is the *G-tightening procedure*:

(1) Let $G_0 = S$. Let $D_0 = D$.
(2) Now do a small normal isotopy of G_0 in the transverse direction while also tightening G_0 along D_0. Call the surface so obtained G_0'. Now surger all simple curves of $f \cap G_0'$ for every $f \subset T^2$ to obtain G_0''. Then delete any bubble components of G_0'' (*i.e.*, two-sphere components which are contained in the interior of tetrahedra). Call the resulting surface G_1. Note that G_1 inherits a transverse orientation from G_0.
(3) At stage $k \geq 1$ there are two possibilities. Suppose first that G_k has an outermost bent arc α with the transverse orientation of G_k pointing into the tightening disk D_k, which is cut out of T^2 by α. Then perform a small normal isotopy of G_k in the transverse direction while tightening G_k across D_k. Call the surface so obtained G_k'. Now surger all simple curves of $f \cap G_k'$ for every $f \in T^2$ to obtain G_k''. Then delete any bubble components of G_k''. Call the resulting surface G_{k+1}. Note that G_{k+1} inherits a transverse orientation from G_k.

    If there is no such outermost bent arc $\alpha \subset G_k$ then set $n = k$ and the procedure halts.

Lemma 13.4. *The surface G_n is normally isotopic to $\widetilde{S}$, the normalization of S.*

Proof. Recall that Lemma 9.1 gives a complete classification of the possible components of intersection of image$(\mathcal{F}_k)$ with the faces of T^2. Again, see Figures 6 and 7. The only components containing a simple curve are the terminal rectangle with hole and terminal hexagon with hole. Hence their names.

Since the terminal with hole rectangles and hexagons do not contain normal or bent arcs of F_k they remain unchanged in the F-tightening procedure until F_n is reached. Then all simple curves are surgered and bubbles capped off. Thus it makes no difference to the resulting surface $\widetilde{S}$ if we surger simple curves and delete bubbles as soon as they appear. □

13.3. **Tightening in I-bundle regions.** We now give the final modification of the tightening procedure. Suppose that $v(S)$ is an almost normal surface vector. Suppose also that S has a transverse orientation pointing at an exceptional tightening disk.

Recall that N_S is the blocked submanifold cut from $|T|$ by the surface S (so that the transverse orientation points into N_S). Also, N_P is the I-bundle part of N_S while $N_C = \overline{N_S \smallsetminus N_P}$ is the core of N_S.

We require slightly more sophisticated data structures. First define product(S) to be the list $\{v_j\}_{j=1}^m$ where the j^{th} element is the vector $2 \cdot v(N_P^j)$ – here $v(N_P^j)$ is the block vector for the j^{th} component of $\widehat{N}_P$, found by Theorem 6.2. That is, $\sum v_j$ counts the normal disks of S which make up the horizontal boundary of the product blocks in N_S.

Put a copy of the horizontal boundary of N_C in core(S). That is, record in core(S) all of the gluing information between edges of disks which are on the horizontal boundary of core blocks. Also record, for each edge in $\partial\partial_v N_C$, which disk of core(S) contains it and which component $N_P^j \subset N_P$ it is glued to. Build a model of N_C. That is, deduce what core blocks occur in which tetrahedra and how they are glued across faces of T^2.

We now turn to constructing a sequence of surfaces H_k. Each H_k will be represented by the two pieces of data: core(H_k) and product(H_k). Here is the *H-tightening procedure*:

(1) Let core(H_0) = core(S). Let product(H_0) = product(S). Let $D_0 = D$. At stage k there is a tightening disk D_k used to alter H_k.
(2) If the D_k has empty intersection with N_P then perform the tightening move as in the G-sequence. This effects only the pieces in core(H_k) and we use the tightening move to compute core(H_{k+1}). Set product(H_{k+1}) = product(H_k) and go to stage $k+1$.
(3) Suppose D_k intersects a component of N_P, say N_P^j. Then set product(H_{k+1}) = product(H_K)$\smallsetminus\{v_j\}$; *i.e.*, remove v_j from the product part. We also alter the disks in the core as follows: Let core(H'_k) = core(H_k) $\cup\, \partial_v N_P^j$. Let $D'_k = D_k \smallsetminus N_P^j$ (that is, remove a small neighborhood of T^1 from D_k). See Figure 12. Then D'_k is a surgery disk for core(H'_k). So surger along D'_k, surger along all simple curves of core(H'_k), and delete all bubbles in core(H'_k). This finally yields core(H_{k+1}). Go on to stage $k+1$.
(4) At stage $k+1$ there are two possibilities: either there is a bent arc in core(H_{k+1}) or there is not. If there is then we have a tightening disk D_{k+1} and proceed as above. If there is no bent arc then sum the vectors in product(H_{k+1}) and add to this vector the number of normal disks of each type in core(H_{k+1}). Output the final sum $v(H_n)$.

This is our final modification of the tightening procedure.

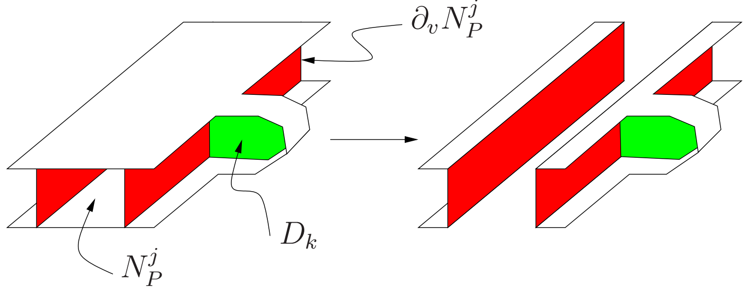

FIGURE 12. Removing the horizontal boundary of N_P^j and adding the vertical.

13.4. **Correctness and efficiency.**

Proof of Theorem 13.1. Note that if the transverse orientation on S points towards an exceptional surgery disk of S then the theorem is trivial. So suppose instead that a tightening disk is pointed at.

Claim. The H-tightening procedure outputs $v(\widetilde{S})$.

That is, we claim that H_n is normally isotopic to $\widetilde{S}$. Since the H procedure is identical to the G procedure in N_C we need only consider the situation where a tightening disk meets N_P. Consider the smallest k such that $D_k \cap N_P \neq \emptyset$. Recall that $\partial D_k = \alpha \cup \beta$ where $\alpha \subset H_k$ and $\beta \subset T^1$. Also by our hypothesis on k the arc β is contained in $T^1 \cap \partial_v \widehat{N}_P$ while only a small neighborhood of $\partial\alpha$ (taken in α) is contained in N_P. Suppose that N_P^j is the component of N_P containing β.

We have assumed that $H_k = G_k$. We will show that we can reorder tightening moves in the G-procedure to obtain $G_{k+k'}$ normally isotopic to H_{k+1}. Then it will follow from Lemma 13.4 that the H procedure produces correct output.

Recall that $\widehat{N}_P^j$ and N_P^j are I-bundles. Let π be the natural quotient map which squashes I-fibres to a point. Let $E = \pi(N_P^j)$. Let $\widehat{E} = \pi(\widehat{N}_P^j)$. Note that $\widehat{E}$ is not necessarily a surface. However E is a surface with boundary, $\widehat{E}$ naturally embeds in E, and there is a small deformation retraction of E to $\widehat{E}$. Note that $\widehat{E}$ and E inherit cell structures from $\widehat{N}_P^j$ and N_P^j. Choose a spanning tree U for the one-skeleton $\widehat{E}^1$ of $\widehat{E}$ rooted at $b = \pi(\beta)$. Choose an ordering of the vertices of U, $\sigma \colon U^0 \to (\mathbb{N} \cap [1, k'])$, so that for any vertex d with parent c we have $\sigma(c) < \sigma(d)$. Here $k' = |U^0|$ is the number of vertices in U^0.

We now have a sequence of tightening moves to perform in the G procedure. At step one do the tightening move along the disk D_k,

surger all simple curves, and delete bubbles. At step $i, i > 1$, examine the edge e between c and d (where $\sigma(d) = i$ and c is the parent of d). Then, by induction and the fact that $\sigma(c) < \sigma(d) = i$ there is a bent arc of G_{k+i-1} in the rectangle $\pi^{-1}(e)$ with endpoints on the segment $\pi^{-1}(d) \subset T^1$. Do this tightening move, surger simple curves, delete bubbles, and go to stage $i + 1$.

After stage $k' = |U^0|$ we obtain the surface $G_{k+k'}$ which is normally isotopic to the following: $(G_k \setminus \partial_h N_P^j) \cup \partial_v N_P^j$ surgered along the disk D'_k, surgered along simple curves, with bubbles deleted. Here $D'_k = \overline{D_k \setminus N_P^j}$. So $G_{k+k'}$ agrees with H_{k+1} and the claim is proved.

Claim. Precomputation for the H procedure takes time at most polynomial in size(T) and log(weight(S)).

Theorem 6.2 computes the vectors $\{v_j\}_{j=1}^m$ in the required amount of time. This gives product(N_S). Then, since there are only a linear number (in size(T)) of core blocks in N_S (Remark 6.1) we can also compute their gluings and so compute core(S) in the alloted time.

Claim. The number of steps in the modified normalization procedure is polynomial in size(T).

Each step either reduces the weight of core(H_k) by two or removes a vector from product(H_k). Since the weight of core(H_k) is at most linear (again, Remark 6.1), and since there are at most a linear number of components of N_P (see Remark 6.3), the claim follows.

Claim. Performing each step of the modified normalization procedure takes time at most polynomial in size(T) and log(weight(S)).

If the tightening disk is disjoint from N_P then we only have to alter core(H_k) in the tetrahedra adjacent to the disk. There are only a linear number of these.

If the tightening disk meets a component of N_P, say N_P^j, then delete v_j from product(H_k) in polynomial time (in size(T)). Alter core(H_k) by gluing on a copy of $\partial_v N_P^j$, surgering along the remnants of the tightening disk D_k, surgering all simple curves, and deleting bubbles. As $\partial_v N_P^j$ is a subset of $\partial_v N_C$ it is at most linear in size (in terms of size(T)). Thus we can make the desired changes in the required time.

To sum up: we can compute the desired result, $v(\widetilde{S})$, in time which is at most a product of polynomials in size(T) and log(weight(S)). This completes the proof of Theorem 13.1. $\square$

14. Crushing, or: "New triangulations for old"

Let T be a triangulation of a closed three-manifold. Suppose we are given a choice of quad type in a single tetrahedron, say the a^{th} type of quad in τ_i. Here $a \in \{1,2,3\}$ and the other two elements of $\{1,2,3\}$ are b and c. Recall that the a^{th} quad type separates the vertices 0 and a from the vertices b and c.

Let θ be the permutation $(0a)(bc)$. Let $\{(i, j_s, \sigma_s)\}_{s=0}^3$ be the four face pairings with i as the first element. Here σ_s glues the s^{th} face of τ_i to some face of τ_{j_s}. Note that $\{(j_s, i, \sigma_s^{-1})\}_{s=0}^3$ are also face pairings in T.

Define a new triangulation T' by *crushing* the tetrahedron τ_i along the a^{th} quad, as follows: Delete τ_i from T. Delete all of the face pairings $\{(i, j_s, \sigma_s)\}_{s=0}^3$. Replace the face pairing (j_s, i, σ_s^{-1}) (if $i \neq j_s$) with

$$\left(j_s, j_{\theta(s)}, \sigma_{\theta(s)} \cdot R_{(s,\theta(s))} \cdot \sigma_s^{-1}\right),$$

for $s \in \{0,1,2,3\}$. Here $R_{(s,\theta(s))}$ is the rotation of the model tetrahedron, about the edge with vertices $\{0,1,2,3\} \smallsetminus \{s, \theta(s)\}$, which takes face s to face $\theta(s)$. Note that $\sigma_{\theta(s)} \cdot R_{(s,\theta(s))} \cdot \sigma_s^{-1}$ is the composition of three orientation reversing maps and thus is also orientation reversing. Finally, no face of any model tetrahedron in T' is glued to itself – thus T' is a triangulation.

To keep track of this operation it may help to refer to the picture of a quad of type 3 shown on the right hand side of Figure 2.

Now suppose that p is a *polarization* of the triangulation T; that is, p is a map from the set of tetrahedra to the set $\{0,1,2,3\}$. Produce a new triangulation T' by *crushing* T along p: To begin with let T' be an exact copy of T. Now, for each $i = 1, 2, \ldots, \text{size}(T)$ do one of two things; If $p(\tau_i) = 0$ simply go on to $i+1$. If $p(\tau_i) \neq 0$ then remove τ_i by crushing along the $p(\tau_i)$ quad, as above, and go on to $i+1$.

The following theorem is now clear:

Theorem 14.1. *There is an polynomial time algorithm which, given a triangulation T and a polarization p, produces T', the triangulation of T crushed along p.* □

Of more interest is:

Theorem 14.2. *Suppose T is a triangulation so that the connect sum $\#|T|$ is a homology three-sphere. Suppose p is a polarization coming from S, a non-vertex linking normal two-sphere. Then the triangulation T', the result of crushing T along p, satisfies $\#|T'| \cong \#|T|$.*

Proof. Theorem 4.10 of Jaco and Rubinstein's paper [13] essentially claims this result for any closed, orientable three-manifold $|T|$ with

the caveat that some connect summands of $|T|$ homeomorphic to lens spaces may by omitted from the crushed $|T'|$. (See also Theorem 3.1 of [2].)

However, by Lemma 4.3 no non-trivial lens space appears as a connect summand of the homology three-sphere $|T|$. Finally, omitting S^3 summands does not change the connect sum. The result follows. □

15. Thompson's Theorem

We will need to use Casson's version [3] of the proof of Thompson's Theorem [18] (which in turn relies heavily on work of Rubinstein [16]):

Theorem 15.1. *There is an exponential time algorithm which, given a triangulation T, decides whether or not $|T|$ is homeomorphic to the three-sphere.*

We now sketch Casson's version of Thompson's algorithm. Begin with a triangulation $T_0 = T$. Check, using Theorems 4.1 and 4.2, that T_0 is a homology three-sphere. Inductively we have a triangulation T_i.

If T_i is not zero-efficient then apply Lemma 5.9 to find $S_i \subset |T_i|$, a fundamental non-vertex-linking normal two-sphere. Let T_{i+1} be the triangulation obtained by crushing along S_i. This requires Theorem 14.1.

If T_i is zero-efficient use Lemma 5.9 to search for almost normal two-spheres. If some component of T_i does not contain an almost normal two-sphere then by Theorem 14.2 and Theorem 5.8 the manifold $|T|$ was not the three-sphere. If S_i is an almost normal two-sphere inside a component T' of T_i then let $T_{i+1} = T_i \smallsetminus T'$.

This completes the algorithm. If T_n is non-empty, then $|T|$ was not the three-sphere. If T_n is empty then $|T|$ was homeomorphic to the three-sphere. Both of these again use Theorem 14.2. This completes our description of Casson's algorithm and our proof of its correctness.

Note that $\mathrm{size}(T_i)+i \leq \mathrm{size}(T)$ as either crushing along a polarization or deleting a component always reduces the number of tetrahedra by at least one. This finishes the sketch of the proof of Theorem 15.1.

16. Showing the problem lies in NP

We are now in a position to prove:

Theorem 16.1. *The three-sphere recognition problem lies in the complexity class* **NP**.

Proof. Suppose that T is a triangulation of the three-sphere. The certificate is a sequence of pairs $(T_i, v(S_i))$ with the following properties.

- $T = T_0$.

- S_i is a normal or almost normal two-sphere, contained in $|T_i|$, with $\mathrm{weight}(S_i) \leq \exp(\mathrm{size}(T_i))$.
- If S_i is normal then S_i is not vertex linking and T_{i+1} is obtained from T_i by crushing along S_i.
- if S_i is almost normal then S_i normalizes to vertex linking two-spheres, in both directions. Also, T_{i+1} is obtained from T_i by deleting the component T' of T_i which contains S_i.
- Finally, the last triangulation T_n is empty, as is S_n.

Note that existence of the certificate is given by our proof of Theorem 15.1. So the only task remaining is to check the certificate. Here we find two subtle points – we will not attempt to verify that the S_i are fundamental nor will we try to check that the T_i containing almost normal two-spheres are zero-efficient.

Note instead, since the S_i are fundamental, they obey the weight bounds given in Lemma 5.5; that is, $\mathrm{weight}(S_i) \leq \exp(\mathrm{size}(T_i))$.

So suppose a certificate $(T_i, v(S_i))$ as above, for the triangulation T, is given to us. First check, using Theorem 4.1 and 4.2, that T is a triangulation of a homology three-sphere.

By Theorem 4.4 check that $T = T_0$. Using Theorem 5.3 verify that S_i is a connected normal or almost normal surface. Using Lemma 5.2 compute the Euler characteristic of S_i. (Here we are using the fact that $\mathrm{weight}(S_i) \leq \exp(\mathrm{size}(T_i))$ in order to compute Euler characteristic in time only polynomial in $\mathrm{size}(T_i)$.) This verifies that S_i is a two-sphere.

If S_i is normal, by Theorem 14.1, crush T_i along S_i in time at most polynomial in $\mathrm{size}(T_i)$. Then check, using Theorem 4.4, that T_{i+1} agrees with the triangulation obtained by crushing T_i.

If S_i is almost normal, we need to check that T', the component of T_i containing S_i, has $|T'| \cong S^3$. Using Theorem 13.1 normalize S_i in both directions in time at most polynomial in $\mathrm{size}(T_i)$ (again, because $\log(\mathrm{weight}(S_i)) \leq \log(\exp(\mathrm{size}(T_i))) = \mathrm{size}(T_i)$). If all components of the two normalizations $\widetilde{S_i^+}$ and $\widetilde{S_i^-}$ are vertex linking two-spheres then T' is a triangulation of the three-sphere, by Theorem 11.3. Finally, use Theorem 4.4 to check that the triangulation $T_i \smallsetminus T'$ is identical to T_{i+1}. ☐

17. Questions and future work

Our techniques should also apply to the following question:

Question. Is the surface bundle recognition problem in **NP**?

Given a triangulation T of a surface bundle the certificate would be a certain normal two-sphere S_0 and collections of surfaces $\{F_i\}$ and $\{G_i\}$.

Here all of the F_i and G_i are fibres of some surface bundle structure on $|T|$, the F_i are all normal, the G_i are all almost normal, G_i normalizes to F_i and F_{i+1} for $i > 0$, and G_0 normalizes to $F_0 \cup S_0$ on one side and to F_1 on the other. (See [17] for some of the necessary existence results.) Also, S_0 bounds a three-ball B_0 in $|T|$ which contains all of the vertices of T. (Note that there is a non-trivial issue here: Corollary 1.1 must be modified to allow us to certify that B_0 is a three-ball.)

Given this it is not unreasonable to ask:

Question. Is the surface bundle recognition problem **NP**-hard?

Perhaps more difficult to resolve would be:

Question. Is the three-sphere recognition problem **NP**-hard?

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